EXERCISE – III

HINTS & SOLUTIONS

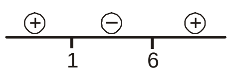
Sol.1
(i) maxima
(ii) minima
(iii) Neither
(iv) Neither
(v) Neither
(vi) maxima

Sol.2 $f(x) = \frac{a}{x} + bx \Rightarrow 6 = a + b$ ......(1)

$f'(x) = -\frac{a}{x^2} + b$

$-a + b = 0 \Rightarrow a = b$
$a = 3,\ b = 3$

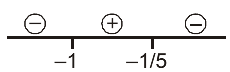
Sol.3 **(i)** $f(x) = 2x^3 - 21x^2 + 36x - 20$
$f'(x) = 6x^2 - 42x + 36$
$= 6(x - 1)(x - 6) = 0$



x = 1 max.

x = 6 min.

(ii) $f(x) = -(x-1)^3 (x+1)^2 = -(x-1)(x^2-1)^2$
$f'(x) = -(x^2-1)^2 - 2(x-1)(x^2-1)(2x)$
$= -(x-1)^2 (x+1)(5x+1)$



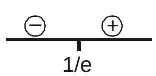
x = – 1 min.

x = – 1/5 max.

(iii) $f(x) = x\,\ell n\,x$
$f'(x) = \ell n\,x + 1 = 0$
$x = 1/e$

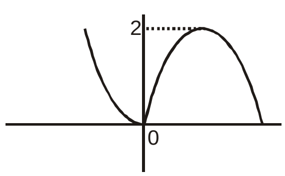
If $x < \frac{1}{e}$

$\ell n\,x < -1$
$\ell n\,x + 1 < 0$



minima at x = 1/e

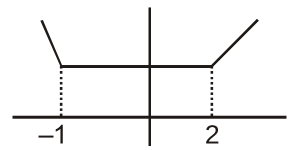
Sol.4



minima at x = 0

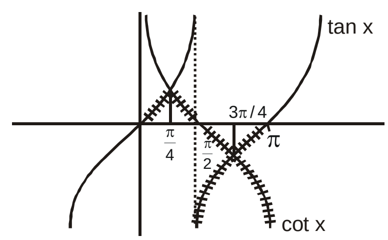
Sol.5 **(i)** $f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105$
$f'(x) = -3x^3 - 24x^2 - 45x$
$= -3x(x^2 + 8x + 15) = -3x(x+3)(x+5)$
critical points are x = 0, –3, –5

(ii) $f(x) = 1 - 2x \quad x < -1$
$= 3 \quad -1 \le x < 2$
$= 2x - 1 \quad x \ge 2$



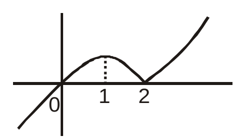
Infinite number of point in [–1, 2]

(iii)



$x = \frac{\pi}{4}$ and $\frac{3\pi}{4}$

Sol.6 $f(x) = x(x-2) \quad x \ge 2$
$= -x(x-2) \quad x < 2$
min at x = 2
max. at x = 1



Sol.7 **(i)** $f(x) = x^3$
absolute max. $= (2)^3 = 8$
absolute min. $= (-2)^3 = -8$

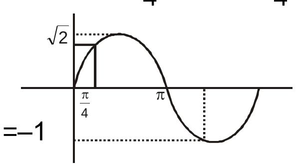
(ii) $f(x) = \sin x + \cos x \quad 0 \le x \le \pi$

$f(x) = \sqrt{2}\,\sin\left(x + \frac{\pi}{4}\right) \quad \frac{\pi}{4} \le x + \frac{\pi}{4} \le x + \frac{\pi}{4}$

$g(t) = \sqrt{2}\,\sin t \quad \frac{\pi}{4} \le t \le x + \frac{\pi}{4}$

absolute max. $= \sqrt{2}$

absolute min. $= \sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) = -1$



(iii) $f(x) = 4x - \frac{x^2}{x}$ [–2, 9/2]

$f'(x) = 4 - x \Rightarrow x = 4$
$f(-2) = -10$

$f(9/2) = \frac{63}{8} \approx 7 \cdot 9$

$f(4) = 8$

absolute max. = 8

absolute min. = – 10

(iv) $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$; $x \in [0, 3]$

$f'(x) = 12(x - 2)(x^2 + 2) = 0$

$x = 2$

$f(0) = 25$

$f(2) = -39$

$f(3) = 16$

Global max. = 25

Global min. = – 39

(v) $f(x) = \sin x + \frac{1}{2} \cos 2x$

$f'(x) = \cos x - \sin 2x = \cos x (1 - 2 \sin x) = 0$

$\cos x = 0$ $\quad \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$

$x = \frac{\pi}{2}$

$f(0) = \frac{1}{2}$

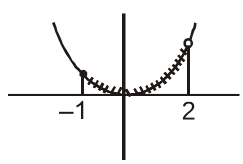
$f\left(\frac{\pi}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2}$ min.

$f\left(\frac{\pi}{6}\right) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ max.

Sol.8 $f(x) = x^2$; $x \in [-1, 2)$

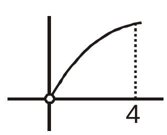
$x = -1$ is a local maxima point

But Global maximum is not defined as $x = 2$ is not included.



Sol.9 $f(x) = x + \sqrt{x}$

$f'(x) = 1 + \frac{1}{2\sqrt{x}} > 0$



for $x \in (0, 4)$

Can't calculate min. & max. value

Sol.10 $f(1) = 1 + \ell n\, b$

$f(1) \le f(1^-)$ $\quad b > 0$

$1 + \ell n\, b \le 2$

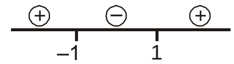
$\ell n\, b \le 1$

$b \le e$

Final answer $b \in (0, e]$

Sol.11 **(i)** $f(x) = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1$



Max. at $x = -1$, $f(-1) = -2$

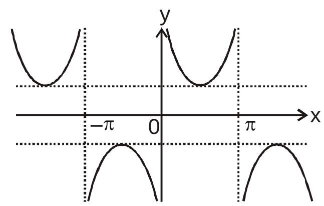
Min. at $x = 1$, $f(1) = 2$

(ii) $f(x) = \text{cosec } x$

minima at

$x = -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ ....

$x = (2n + 1)\frac{\pi}{2}$ $\quad n \in I$



maxima at $x = -\frac{\pi}{2}, \frac{3\pi}{2}$ ....

$x = (2n - 1)\frac{\pi}{2}, \quad n \in I$

$f_{max.} = 1, \; f_{min.} = -1$

Sol.12 $y = \sin^p \theta \quad \cos^q \theta \quad p, q \in N$

Let $z = \ell n\, y = p\, \ell n \sin \theta + q\, \ell n \cos \theta$

$\frac{dz}{d\theta} = \frac{p}{\sin\theta} \cos \theta - \frac{q}{\cos\theta} \sin \theta = 0$

$p \cot \theta - q \tan \theta = 0$

$\tan^2 \theta = p/q$

$\theta = \tan^{-1} \sqrt{\frac{p}{q}}$

$\frac{d^2z}{d\theta^2} = -p\, \text{cosec}^2\theta - q \sec^2\theta < 0$ when $\tan^2 \theta = p/q$

so maxima when $\theta = \tan^{-1} \sqrt{\frac{p}{q}}$

Sol.13 Gives AB + AC = constant = k

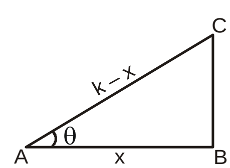
If AB = x $\Rightarrow$ AC = k – x

$BC^2 = (AC)^2 - AB^2$

$= (k - x)^2 - x^2 = k^2 - 2kx$

$\Delta = \frac{1}{2}$ (BC) × (AB)

$\Delta = \frac{1}{2} \times \sqrt{k^2 - 2kx}$



$z = \Delta^2 = \frac{1}{4} x^2 (k^2 - 2kx)$

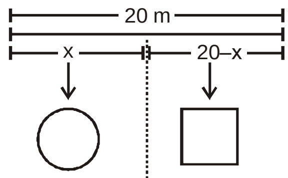
$\frac{dz}{dx} = \frac{1}{4}[2x(k^2 - 2kx) + x^2(-2k)] = 0$

$x = 0, \quad x = \frac{k}{3}$

$\cos \theta = \frac{x}{k - x} = \frac{k/3}{k - k/3} = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

Sol.14



2pr = x   4a = 2a – x

$r = \frac{x}{2x}$ $a = \frac{20-x}{4}$

$A_1 = \frac{x^2}{4x}$ $A_2 = \left(\frac{20-x}{4}\right)^2$

$A = A_1 + A_2$

$$A = \frac{x^2}{4\pi} + \frac{(20-x)^2}{4}$$

$$\frac{dA}{dx} = \frac{2x}{4\pi} - \frac{2(20-x)}{16} = 0 \Rightarrow x = \frac{20\pi}{4+\pi}$$

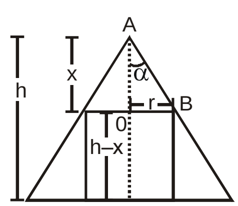
$$\frac{d^2A}{dx^2} = \frac{2}{4\pi} + \frac{2}{16} > 0$$

Area will be minimum

$$20 - x = 20 - \frac{20\pi}{4+\pi} = \frac{80\pi}{4+\pi}$$

Length are $\frac{20x}{4+x}$ & $\frac{80x}{4+x}$

Sol.15 In ΔAOB



$\tan \alpha = \frac{r}{x}$

$r = x \tan \alpha$ ......(1)

$v = \pi r^2 H$

$= \pi r^2 (h - x)$

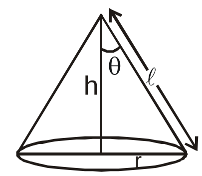
$v = \pi x^2 \tan^2 \alpha (h - x)$

$$\frac{dv}{dx} = 0 \Rightarrow x = \frac{2h}{3}$$

$$v = \pi \times \left(\frac{2h}{3}\right)^2 \times \tan^2 \alpha \left(h - \frac{2h}{3}\right)$$

$$v = \frac{4}{27} \pi h^3 \tan^2 \alpha$$

Sol.16 $v = \frac{1}{3}\pi r^2 h$



$r = \ell \sin \theta$

$h = \ell \cos \theta$

$$v = \frac{1}{3}\pi \ell^3 \sin^2\theta \cos \theta$$

$$= \frac{1}{3}\pi \ell^3 (1 - \cos^2\theta) \cos \theta$$

$$v = \frac{1}{3}\pi \ell^3 (\cos \theta - \cos^3\theta)$$

$$\frac{dv}{d\theta} = \frac{1}{3}\pi \ell^3 (-\sin \theta + 3\cos^2 \theta \sin \theta)$$

$\sin \theta = 0$ or $3\cos^2 \theta = 1$

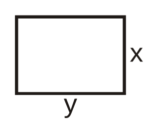
(reject)

$\cos \theta = \frac{1}{\sqrt{3}}$

$\tan \theta = \sqrt{2}$

$\theta = \tan^{-1} \sqrt{2}$

Sol.17 Given that



$2(x + y) = 36$

$\Rightarrow x + y = 18$

when revolved around one of its side (y) then it will turn out into cylinder whose height y and radius = x

$v = \pi r^2 h = \pi x^2 y$

$v = \pi x^2(18 - x) = \pi(18x^2 - x^3)$

$$\frac{dv}{dx} = \pi(36x - 3x^2) = 0 \Rightarrow x = 12$$

$$\frac{d^2v}{dx^2} = \pi(36 - 6x) < 0$$

max. at x = 12

x = 12, y = 6

Sol.18 Let speed of train = V m/H

covering distance = S

Time taken to couver distance = S/V Hour

fuel charges = kV^2 ($k \rightarrow$ constant)

$48 = k \cdot (16)^2 \Rightarrow k = 3/16$

other cost (per hours) = 300 Rs

Per hour total cost = $\frac{3}{16}V^2 + 300$

For a run total cost

$$E = \left(\frac{3}{16}V^2 + 300\right)\frac{S}{V}$$

$$E = S\left(\frac{3}{16}V + \frac{300}{V}\right)$$

$$\frac{dE}{dV} = S\left(\frac{3}{16} - \frac{300}{V^2}\right) = 0$$

$V^2 = 1600 \Rightarrow V = 40$

$$\frac{d^2E}{dV^2} = S\left(\frac{600}{V^3}\right) > 0 \quad \text{for } V = 40$$

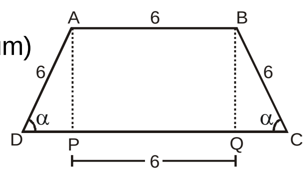
E is minimum

most economical speed is 40 m/H

Sol.19 AP = BQ = 6 sin α

DP = QC = 6 cos α

A = (Area of trapezium)



$$= \frac{1}{2}[AB + DC] \,.\, BQ$$

$$= \frac{1}{2}[6 + 6 + 12\cos\alpha] \,.\, 6\sin\alpha$$

$$A = 36\left(\sin\alpha + \frac{1}{2}\sin 2\alpha\right)$$

$$\frac{dA}{d\alpha} = 36(\cos\alpha + \cos 2\alpha) = 0$$

$\cos 2\alpha = -\cos\alpha = \cos(\pi - \alpha)$

$3\alpha = \pi \Rightarrow \alpha = \frac{\pi}{3}$

$$\frac{d^2A}{d\alpha^2} = 36(-\sin\alpha - 2\sin 2\alpha) < 0 \text{ as } \alpha = \frac{\pi}{3}$$

A is maximum when $\alpha = \frac{\pi}{3}$

$$A_{max} = 36\left[\sin 60^\circ + \frac{1}{2}\sin 120^\circ\right]$$

$= 27\sqrt{3}$ square cm

Sol.20 Let length = x

width = y

Given $xy = 18$ ......(1)

After taking margin

New length = x – 3/4

New width = y – 1/2

New area = $\left(x - \frac{3}{4}\right)(y - 1/2)$

$$A = xy - \frac{x}{2} - \frac{3}{4}y + \frac{3}{8}$$

$$A = 18 - \frac{x}{2} - \frac{3}{4} \times \left(\frac{18}{x}\right)$$

$$\frac{dA}{dx} = -\frac{1}{2} + \frac{3\times 9}{2}x^2 = 0$$

$\Rightarrow x^2 = 27 \Rightarrow x = 3\sqrt{3}$; $y = 2\sqrt{3}$

Sol.21 $f'(x) = ax^2 + 2(a + 2)x + (a - 1)$

Let $g(x) = ax^2 + 2(a + 2)x + (a - 1)$

g(x) must have both roots negative

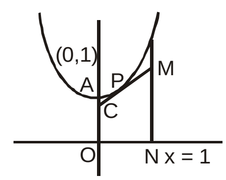
$D > 0 \Rightarrow a > -4/5$

$g(0) > 0 \Rightarrow a > 1$

$-\frac{b}{2a} < 0 \Rightarrow a < -2, a > 0$

Finally $a \in (1, \infty)$

Sol.22 Let point $P(h, 1 + h^2)$



$$\frac{dy}{dx} = 2x = 2h$$

Tangent at P :

$y - (1 + h^2) = 2h(x - h)$

For point L put x = 0

$y = 1 - h^2$ $\qquad$ $L(0, 1 - h^2)$

For point M put x = 1

$M(1, 1 + 2h - h^2)$, $N(1, 0)$

A = area of trapezium = $\frac{1}{2}(OL + MN)\cdot ON$

$$= \frac{1}{2}[1 - h^2 + 1 + 2h - h^2].1$$

$A = 1 + h - h^2$

$$\frac{dA}{dh} = 1 - 2h = 0 \qquad \Rightarrow h = 1/2$$

$$\frac{d^2A}{dh^2} = -2 < 0$$

$$P\left(\frac{1}{2}, \frac{5}{4}\right)$$

Sol.23 $f(x) = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$

Given that $\lim_{x\to 0}\left(1 + \frac{f(x)}{x^3}\right)^{1/x} = e^2$, 1^∞ Form

For existance of limit $\quad$ e,f,g = 0

$f(x) = ax^6 + bx^5 + cx^4 + dx^3$

$$\lim_{x\to 0} \frac{f(x)}{x^3} = ax^3 + bx^2 + cx + d \to 0$$

then d = 0

$f(x) = ax^6 + bx^5 + cx^4$

$f'(x) = 6ax^5 + 5bx^4 + 4cx^3$

$= x^3 (6ax^2 + 5bx + 4c)$ .....(1)

$$\lim_{x\to 0} \left(1+\frac{f(x)}{x^3}\right)^{1/x} = e^2$$

$$\lim_{x\to 0} \frac{f(x)}{x^4} = 2 \Rightarrow c = 2$$

$f'(x) = x^3 (6ax^2 + 5bx + 8)$

$f'(1) = f'(2) = 0$

$f'(1) = 0 \Rightarrow 6a + 5b + 8 = 0$

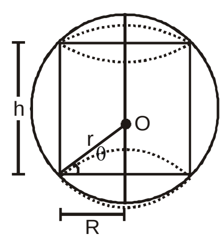
$f'(2) = 0 \Rightarrow 24a + 10b + 8 = 0$ By solving

$$a = \frac{2}{3}, \ b = -\frac{12}{5}$$

so $f(x) = \frac{2}{3}x^6 - \frac{12}{5}x^5 + 2x^4$

Sol.24 Let radius and height be R and H

By figure



$R = r\cos\theta$

$H = 2r\sin\theta$

$v = \pi R^2 H$

$= \pi r^2 \cos^2\theta (2r\sin\theta)$

$v = 2\pi r^3 (\cos^2\theta \sin\theta)$

$$\frac{dv}{d\theta} = 2\pi r^3 [-2\cos\theta \sin^2\theta + \cos^2\theta\cos\theta]$$

$-2\cos\theta(1-\cos^2\theta) + \cos^3\theta = 0$

$-2\cos\theta + 2\cos^3\theta + \cos^3\theta = 0$

$3\cos^3\theta = 2\cos\theta$

$\cos\theta = 0$ (reject) or $\cos\theta = \sqrt{\frac{2}{3}}$

$$\sin\theta = \frac{1}{\sqrt{3}}$$

$$v_{max} = 2\pi r^3 \left(\frac{2}{3}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}$$

Sol.25 If annual increase subscribtion = x Rs

Then rate will be = (300 + x) Rs

and subcribers = (500 – x)

New income I = (300 + x) (500 – x)

$I = 150000 + 200x - x^2$

$$\frac{dI}{dx} = 200 - 2x = 0 \Rightarrow x = 100$$

$$\frac{d^2I}{dx^2} = -2 < 0$$

x = 100 will be maximum point

charge per scriber = (300 + x)

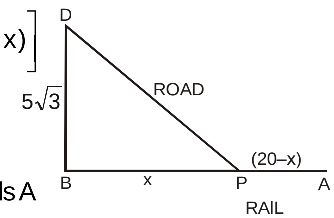
= (300 + 100) = 400 Rs.

Sol.26 Let the freight charges of rail = R

then freight charges of road = 2R

If total cost = c

$$c = R\left[2\sqrt{x^2+75} + (20-x)\right]$$

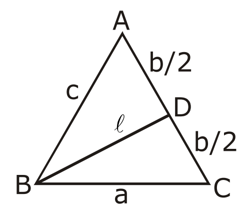


$$\frac{dc}{dx} = 0 \Rightarrow x = 5$$

5 km from B towards A

Sol.27 b = c

In ΔABD



$$\cos\theta = \frac{c^2 + b^2/4 - \ell^2}{bc}$$

$$\cos\theta = \frac{5b^2 - 4\ell^2}{4b^2} \quad ...(i)$$

$A = 2 \times \frac{1}{2} b^2 \sin\theta = b^2\sin\theta$

$A^2 = b^4 \sin^2\theta$

$$= b^4 \left[1 - \left(\frac{5b^2 - 4\ell^2}{4b^2}\right)^2\right]$$

$$A^4 = b^4 \ \frac{\left(5b^2 - 4\ell^2\right)^2}{16}$$

Diff. w.r.t. b

and put the value of b^2 in (i)

Sol.28 $y = 1 - x^2$

Let point $P(x_0, 1 - x_0^2)$

$$\frac{dy}{dx} = -2x\Big|_p = -2x_0$$

tangent at p

$y - (1 - x_0^2) = -2x_0 (x - x_0)$

For point A put y = 0

$$x = \frac{x_0^2 + 1}{2x_0}$$

For point B put x = 0 $\Rightarrow$ $y = x_0^2 + 1$

A = area of Δ OAB = $\frac{1}{2}\,\frac{\left(x_0^2+1\right)}{2x_0}\left(x_0^2+1\right)$

$$A = \frac{1}{4}\left[\frac{x_0^4+2x_0^2+1}{2x_0}\right] = \frac{1}{4}\left[x_0^3+2x_0+\frac{1}{x_0}\right]$$

$$\frac{dA}{dx_0} = \frac{1}{4}\left[3x_0^2+2-\frac{1}{x_0^2}\right]$$

$$\frac{dA}{dx_0} = 0 \Rightarrow 3{x_0}^2 + 2 - \frac{1}{x_0^2} = 0$$

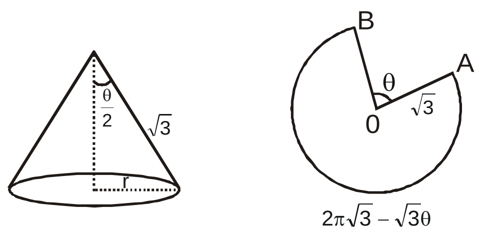
$${x_0}^2 = \frac{1}{3} \quad \text{or} \quad {x_0}^2 = -1 \ \text{(reject)}$$

$$x_0 = \frac{1}{\sqrt{3}} \qquad x_0 = -\frac{1}{\sqrt{3}} \ \text{(reject)}$$

$$A = \frac{1}{4}\,\frac{(x_0^2+1)^2}{x_0}$$

$$A_{max} = \frac{1}{4}\left(\frac{\left(1+\frac{1}{3}\right)^2}{1/\sqrt{3}}\right) = \frac{4\sqrt{3}}{9}$$

Sol.29



$$h = \sqrt{3}\ \cos\frac{\theta}{2}$$

$$r = \left(\frac{2\pi\sqrt{3}-\sqrt{3}\theta}{2x}\right)$$

$$v = \frac{1}{3}\pi r^2 h$$

$$v = \frac{1}{3}\pi\left(\frac{2\pi\sqrt{3}-\sqrt{3}\theta}{2x}\right)^2 \cdot \sqrt{3}\ \cos\frac{\theta}{2}$$

$$\frac{dv}{d\theta} = 0 \Rightarrow \theta = \frac{2\pi}{3}$$

Sol.30 Let base = x, height = h

given $x^2 h = 1000$

upper part = x^2

base = x^2 ; hypoteneous part = 4xh

Total cost

$$C = 15x^2 + 25x^2 + 20(4xh) + 300$$

$$= 40x^2 + 80x\left(\frac{1000}{x^2}\right) + 300$$

$$\frac{dc}{dx} = 0 \Rightarrow 80x - \frac{80\times1000}{x^2} = 0$$

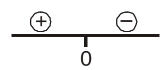
$$x^3 = 1000 \qquad \Rightarrow x = 10$$

$$h = \frac{1000}{x^2} = 10$$

Base = 10, height = 10

Sol.31 $f(x) = \tan^{-1}x - \frac{1}{2}\,\ell n\, x \qquad D_f : x > 0$

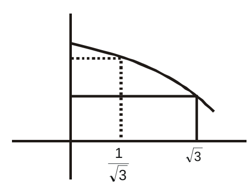
$$f'(x) = \frac{1}{1+x^2} - \frac{1}{2}\,\frac{1}{x} = \frac{2x-(1+x^2)}{2x(1+x^2)} = -\frac{(x-1)^2}{2x(1+x^2)}$$



f(x) is decreasing for $\forall$ x > 0

Global maxima at x = $\frac{1}{\sqrt{3}}$

$f\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} + \frac{1}{4}\,\ell n\, 3$



Global minima at x = $\sqrt{3}$

$f(\sqrt{3}) = \frac{\pi}{3} - \frac{1}{4}\,\ell n\, 3$

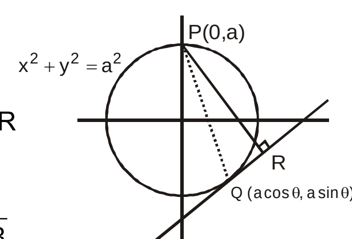
Sol.32 Tangent x cos θ + y sin θ = a

PR = a(1 – sin θ)

PQ = distance formula

Then by $PQ^2 = PR^2 + QR^2$

$QR = \sqrt{PQ^2 - PR^2}$



Area = $\frac{1}{2}$ × PR × QR

= f(θ)

$$\frac{dA}{d\theta} = 0 \Rightarrow A = \frac{3\sqrt{3}}{8}a^2$$